LETTER TO THE EDITOR

Biased percolation: forest fires with wind

T Ohtsuki and T Keyes

Department of Chemistry, Boston University, Boston, MA 02215, USA

Abstract. Non-equilibrium percolating–non-percolating phase transitions in kinetic growth of percolation clusters in the presence of an external bias (biased percolation) describing the spread of forest fires with wind, etc, are investigated theoretically. A real space renormalisation group technique is used to calculate a phase diagram, critical exponents and spreading velocities explicitly in a square lattice. The effects of the bias on phase transitions and critical phenomena are clarified. The bias causes the directed spread belonging to a different universality class from that of the unbiased process. New critical exponents for spreading velocities of directed and reverse directed percolation processes are introduced and evaluated explicitly.

Recently, there has been a growing interest in the investigation of kinetic growth of random clusters. Some processes exhibit a percolating–non-percolating phase transition and build up percolation clusters (Alexandrowicz 1980). For instance, a model of forest fires studied by MacKay and Jan (1984) belongs to the same universality class as that of usual bond percolation (Janssen 1985, Cardy and Grassberger 1985). This model also describes autocatalytic chemical reactions and epidemic processes with immunisation (Grassberger 1983). In addition, it provides a typical example of many subjects of much current interest such as applications of percolation theories to dynamic processes (see, e.g., Wilkinson and Willemsen 1983, Kerstein 1984, Schulman and Seiden 1983), kinetic growth of fractal objects (Family and Landau 1984), non-equilibrium phase transitions and pattern formation (Nicolis and Prigogine 1977, Haken 1978) and (stochastic) cellular automata (Wolfram 1983, Kinzel 1985). Then the study of these processes is considered to be quite significant both practically and theoretically.

MacKay and Jan (1984) pointed out the importance of an external bias due to wind, topography, etc, in the spread of forest fires. In chemical reactions and epidemic processes, external fields, such as solvent flow, are thought to play an important role, too. In thermal and geometrical (percolative) critical phenomena, it is known that, in the vicinity of critical points, systems show singular responses to external perturbations (Onuki and Kawasaki 1979, Meron and Procaccia 1983, Ohtsuki and Keyes 1984a). In kinetic growth processes, however, little is known about the influence of external fields. The purpose of this letter is to investigate biased growth of percolation clusters (biased percolation) and to clarify the effects of the bias on phase transitions and critical phenomena. We adopt a real space renormalisation group (RSRG) method and make explicit calculations of a phase diagram, critical exponents and spreading velocities.

We consider the following process in a square lattice. Here we use the terminology of forest fires. The system is composed of fresh (unburnt), burning and burnt trees

(sites). We start with one burning tree in a densely packed lattice of fresh trees and construct a cluster of burnt trees step by step. Burning trees ignite their nearest-neighbour fresh trees with some ignition probability. A same bias (wind) is applied along both axes of a square lattice. Then ignition probabilities p_+ in two preferred directions are equal and larger than those p_- in opposite directions. At the end of each step, all previously burning trees are turned into burnt trees and never ignited again. Note that when $p_+ = p_-$, $p_- = 0$ or $p_+ = 1$, the process belongs to the same universality class as that of isotropic, directed (Kinzel 1983) or reverse directed (Redner 1983) percolation, respectively.

Hereafter, we assume the existence of a RSRG transformation. This assumption is justified by self-similarity of percolation clusters (Kapitulnik *et al* 1983). First, we determine recursion relations g_+ and g_- for p_+ and p_- and compute a phase diagram. A usual cell-to-cell decimation scheme with a rescaling factor $b = 2$ is adopted and a group of bonds and sites are combined into superbonds and supersites, as illustrated in figure 1 (Reynolds *et al* 1977, Stanley *et al* 1982). Both on the original and the
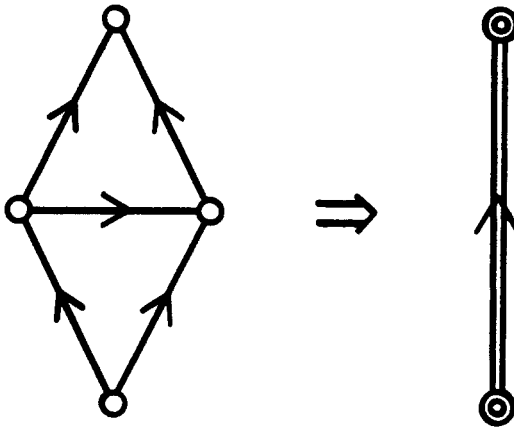

Figure 1. Cell-to-cell decimation scheme.

renormalised cell, the process is advanced until the top site is ignited or burning trees become extinct after starting with a configuration where only the origin (bottom site) is burning and others are fresh. Then ignition probabilities of the top site are calculated on both cells. In the spirit of a RSRG approach, the ignition probability is kept invariant under the transformation. Equating calculated probabilities, therefore, we obtain g_+ and g_-:

$$p'_+ = g_+(p_+, p_-) = 2p_+^2 + p_+^3 - 3p_+^4 + p_+^5 + p_-(p_+^2 - 2p_+^3 + p_+^4) \tag{1}$$

$$p'_- = g_-(p_+, p_-) = 2p_-^2 + p_-^3 - 3p_-^4 + p_-^5 + p_+(p_-^2 - 2p_-^3 + p_-^4) \tag{2}$$

where a prime denotes a renormalised quantity.

Recursion relations (1) and (2) lead to a p_+ against p_- phase diagram shown in figure 2. There exist three phases. The phase I is a non-percolating phase where both p_+ and p_- go to zero and only finite spread of the fire occurs. The phase III is an isotropic percolating phase where both p_+ and p_- go to unity and there is a non-zero probability of infinite spread of the fire in all directions. The intermediate phase II is a directed percolating phase where $p_+ \to 1$, $p_- \to 0$ and the fire can spread infinitely only

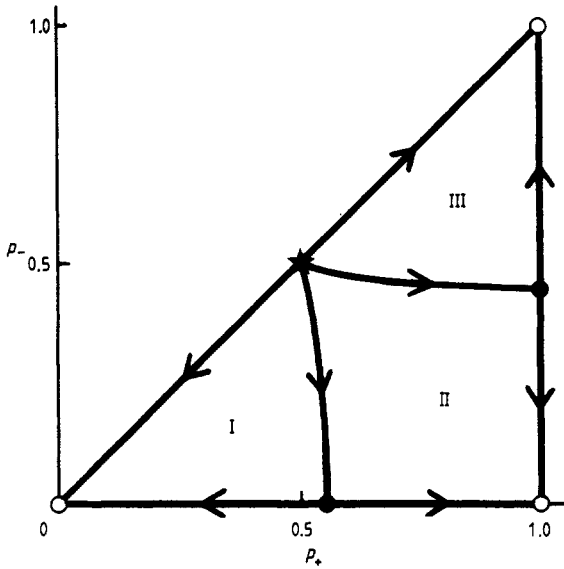**Figure 2.** p_+ against p_- phase diagram. The star ($\star$) and solid circles ($\bullet$) denote non-trivial fixed points.

within a certain opening angle ϕ. On the boundary between the phase, I and II, $\phi = 0$, while on the II–III boundary, $\phi = \pi/2$ (Kinzel 1983, Redner 1983). Between these boundaries (in the phase II), ϕ is considered to vary continuously from 0 to $\pi/2$.

From (1) and (2), we can also derive critical exponents ν for a correlation length ξ (Stauffer 1985, Reynolds *et al* 1977, Stanley *et al* 1982). There are three non-trivial fixed points: (*a*) $p_+^* = p_-^* = 0.5$, (*b*) $p_+^* = 0.555$, $p_-^* = 0$ and (*c*) $p_+^* = 1$, $p_-^* = 0.445$. As mentioned before, the process along the line $p_+ = p_-$, $p_- = 0$ or $p_+ = 1$ is equivalent to isotropic, directed (Kinzel 1983) or reverse directed (Redner 1983) percolation. Thus, fixed points (*a*), (*b*) and (*c*) describe corresponding percolation thresholds. The isotropic percolation threshold (*a*) is a tricritical point and in the vicinity of (*a*), ξ obeys

$$\xi(\varepsilon_1, \varepsilon_2) = \varepsilon_1^{-\nu_1} F_\xi(\varepsilon_1^{\nu_1}\varepsilon_2^{-\nu_B}) \tag{3}$$

$$\xi(\varepsilon_1, 0) \propto \varepsilon_1^{-\nu_1} \tag{4}$$

$$\xi(0, \varepsilon_2) \propto \varepsilon_2^{-\nu_B} \tag{5}$$

where F_ξ is a scaling function, $\varepsilon_1 \equiv |\varepsilon_+ + \varepsilon_-|$, $\varepsilon_2 \equiv |\varepsilon_+ - \varepsilon_-|$, $\varepsilon_+ \equiv |p_+ - p_+^*|/p_+^* \ll 1$ and $\varepsilon_- \equiv |p_- - p_-^*|/p_-^* \ll 1$. Critical exponents are given by $\nu_1 = 1.43$ and $\nu_B = 1.71$. The behaviour (4) appears along the line $p_+ = p_-$ and is the same as that of isotropic percolation, whereas the behaviour (5) is observed when the threshold is approached via the phase II. Critical exponents around (*b*) and (*c*) become $\nu_D = 1.54$ and $\nu_R = 1.54$, where ν_D and ν_R are defined by $\xi(\varepsilon_+) \propto \varepsilon_+^{-\nu_D}(p_- = 0)$ and $\xi(\varepsilon_-) \propto \varepsilon_-^{-\nu_R}$ ($p_+ = 1$). The exponent ν_D is equal to that $\nu_\parallel$ for a parallel correlation length of directed percolation and the value 1.54 is a little smaller than that (1.76) of known estimates (Kinzel 1983).

Transforming p_+ and p_- into $p_0 = (p_+ + p_-)/2$ and $B = p_+ - p_-$, we can regard figure 2 as a p_0 against B phase diagram, where p_0 represents the average ignition probability and B measures the strength of the bias (wind). Here the base line $p_+ = p_-$ expresses the p_0 axis ($B = 0$) and the distance from this line stands for B. Hence, the phase

diagram informs us that generally the bias causes qualitative change of the process, i.e. the behaviour belonging to the different universality class (the directed percolating phase II). As approaching the threshold $p_0^* = 0.5$, the critical value B^* of the bias for this change goes to zero and just at $p_0^* = 0.5$, in particular, even an infinitesimally small bias gives rise to the directed spread. From (3), we find

$$B^* \propto \varepsilon_0^{\nu_1/\nu_B} \tag{6}$$

where $\varepsilon_0 \equiv |p_0 - p_0^*|/p_0^* \ll 1$. It becomes evident that the biased growth of percolation clusters provides another example of singular responses to external fields near critical points (Onuki and Kawasaki 1979, Meron and Procaccia 1983, Ohtsuki and Keyes 1984a).

Next, we calculate the spreading velocity V of directed ($p_- = 0$) and reverse directed ($p_+ = 1$) percolation processes. The velocity V of the isotropic ($p_+ = p_-$) percolation process will be reported elsewhere. We apply a formalism quite similar to that for a diffusion coefficient on percolation lattices developed before (Ohtsuki and Keyes 1984b). Here V is a function of three parameters: a lattice constant l, an ignition probability p and a unit time w of the process necessary to advance one step, where p denotes p_+ (directed process) or p_- (reverse directed process). In order to determine V, therefore, a recursion relation f for w is necessary. This time, we calculate an average step number n until the top site is ignited and put $w'n' = wn$, which means preservation of a real time until ignition of the top site under the transformation. The resulting recursion relations are

$$w'/w = f_+(p_+) = (4 + 3p_+ - 8p_+^2 + 3p_+^3)/(2 + p_+ - 3p_+^2 + p_+^3) \qquad (p_- = 0) \tag{7}$$

$$w'/w = f_-(p_-) = (7 - 3p_- - 5p_-^2 + 3p_-^3)/(3 - p_- - 2p_-^2 + p_-^3) \qquad (p_+ = 1). \tag{8}$$

The π theorem (Barenblatt 1979) and the requirement that V is kept invariant under the transformation lead to

$$V(l', p', w'; t) = V(l, p, w; t) = lw^{-1}V^*(p, T) \tag{9}$$

where $T \equiv t/w$ and V^* is a normalised dimensionless velocity.

Substitution of (1), (2), (7), (8) and the relation $l' = bl$ into (9) yields a recursion relation for V^*:

$$V^*(p, T) = bf^{-1}V^*(g, f^{-1}T) \tag{10}$$

where g, f are g_+, f_+ or g_-, f_-. Iterating the transformation k times, we have

$$V^*(p, T) = p_k \prod_{i=0}^{k-1} b/f(p_i)$$

$$p_{i+1} = g(p_i) \qquad (p_0 = p) \tag{11}$$

$$T = T_k = \prod_{i=0}^{k-1} f(p_i).$$

In this letter, V is defined by a time derivative of an average cluster size, e.g. a radius of gyration. Thus, we use an initial condition $V(l, p, w; w) = plw^{-1}$, because at $t = w$, namely after one step, a cluster spreads pl on average. Using (11), we can perform explicit calculations of V^*. In general, V^* depends on the opening angle ϕ. In the directed percolation process ($p_- = 0$), V_+^* obtained from (11) describes that at $\phi = 0$, while in the reverse process ($p_+ = 1$), V_-^* calculated from (11) represents that at $\phi = \pi$.

Scaling relations for V^* in the vicinity of a threshold as $\varepsilon \equiv |p - p^*|/p^* \ll 1$ are also derived from (10). Linearising recursion relations g and f around fixed points and substituting them into (10), we get

$$V^*(\varepsilon, T) = b\lambda_w^{-1} V^*(\lambda_p \varepsilon, \lambda_w^{-1} T) \tag{12}$$

where $\lambda_p = \partial g/\partial p|_{p=p^*}$ and $\lambda_w = f|_{p=p^*}$. It follows that V^* is a generalised homogeneous function and satisfies scaling relations (Stanley 1971)

$$V^*(\varepsilon, T) = \varepsilon^\theta F_V(\varepsilon T^{\chi/\nu}) \tag{13}$$

$$V^*(\varepsilon, \infty) \propto \varepsilon^\theta \tag{14}$$

$$V^*(0, T) \propto T^{\chi - 1} \tag{15}$$

with critical exponents $\theta \equiv \ln(\lambda_w/b)/\ln(\lambda_p)$, $\chi \equiv \ln(b)/\ln(\lambda_w)$ and $\nu \equiv \ln(b)/\ln(\lambda_p)$. The relaxation time τ of the process is estimated from $\xi \propto \varepsilon^{-\nu} \sim V\tau \propto \tau^\chi$, where $V\tau$ represents an average cluster size. Hence, the following relation among critical exponents hold:

$$\psi = \theta + \nu = \nu/\chi \tag{16}$$

where ψ is a critical exponent for τ defined by $\tau \propto \varepsilon^{-\psi}$ (Alexandrowicz 1980, Kinzel 1983). Geometrically, ψ describes the length of a minimum path (Pike and Stanley 1981) along a directed or reverse directed percolation cluster. Notice that θ, or equivalently χ or ψ, is a new critical exponent introduced here for the first time.

Recursion relations (1), (2), (7) and (8) give explicit values of critical exponents $\theta_D = 0.07$, $\chi_D = 0.96$, $\psi_D = 1.61$, $\theta_R = 0.21$, $\chi_R = 0.88$ and $\psi_R = 1.75$. The calculated results of V^* and F_V are plotted in figures 3 and 4. Figure 3 shows asymptotic values $V^*(p, \infty)$. Small values of the exponents θ cause a sharp increase of $V^*(p, \infty)$ near p^*. The
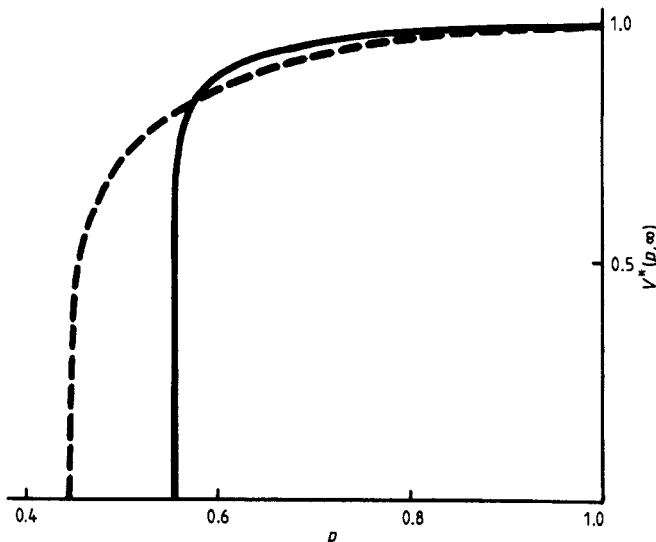

Figure 3. Asymptotic values of the spreading velocity $V^*(p, \infty)$ for the directed (——) and reverse directed (- - - - -) percolation processes.
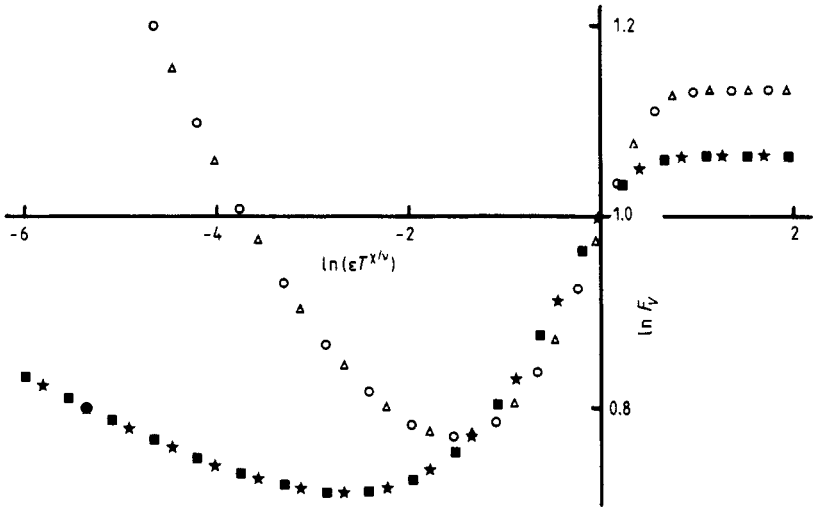
Figure 4. Scaling function F_V for the directed percolation process at $\varepsilon_+ = 0.001$ ($\star$) and 0.0002 ($\blacksquare$) and that for the reverse directed percolation process at $\varepsilon_- = 0.001$ ($\bigcirc$) and 0.0002 ($\triangle$).

scaling functions F_V are plotted in figure 4. The scaling relation holds well and the crossover occurs smoothly.

References

Alexandrowicz Z 1980 *Phys. Lett.* **80A** 284
Barenblatt G I 1979 *Similarity, Self-Similarity and Intermediate Asymptotics* (New York: Consultants Bureau)
Cardy J L and Grassberger P 1985 *J. Phys. A: Math. Gen.* **18** L267
Family F and Landau D P 1984 *Kinetics of Aggregation and Gelation* (Amsterdam: North-Holland)
Grassberger P 1983 *Math. Biosci.* **63** 157
Haken H 1978 *Synergetics* (Berlin: Springer)
Janssen H K 1985 *Z. Phys.* B **58** 311
Kapitulnik A, Aharony A, Deutscher G and Stauffer D 1983 *J. Phys. A: Math. Gen.* **16** L269
Kerstein A R 1984 *Phys. Rev.* B **30** 2980
Kinzel W 1983 *Ann. Israel Phys. Soc.* **5** 425
—— 1985 *Z. Phys.* B **58** 229
MacKay G and Jan N 1984 *J. Phys. A: Math. Gen.* **17** L757
Meron E and Procaccia I 1983 *Phys. Rev. Lett.* **51** 15
Nicolis G and Prigogine I 1977 *Self-Organization in Nonequilibrium Systems* (New York: Wiley)
Ohtsuki T and Keyes T 1984a *Phys. Rev. Lett.* **52** 1177
—— 1984b *Phys. Lett.* **104A** 77
Onuki A and Kawasaki K 1979 *Ann. Phys., NY* **121** 456
Pike R and Stanley H E 1981 *J. Phys. A: Math. Gen.* **14** L169
Redner S 1983 *Ann. Israel Phys. Soc.* **5** 447
Reynolds P J, Klein W and Stanley H E 1977 *J. Phys. C: Solid State Phys.* **10** L167
Schulman L S and Seiden P E 1983 *Ann. Israel Phys. Soc.* **5** 251
Stanley H E 1971 *Introduction to Phase Transitions and Critical Phenomena* (Oxford: Clarendon)

Stanley H E, Reynolds P J, Redner S and Family F 1982 *Real-Space Renormalization* ed T W Burkhardt and J M J van Leeuwen (Berlin: Springer) p169
Stauffer D 1985 *Introduction to Percolation Theory* (London: Taylor and Francis)
Wilkinson D and Willemsen J F 1983 *J. Phys. A: Math. Gen.* **16** 3365
Wolfram S 1983 *Rev. Mod. Phys.* **55** 601